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The Closed Socle of a Central Separable Algebra

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INTRODUCTION

Let R be an integral domain and A a central separable R-algebra. The purpose of this paper is to introduce an ideal of R, the "closed socle" of the title, which is connected with A in a natural way and whose properties have interesting consequences for the algebra A. The main result (Theorem 4.6) is the following: Suppose R is a local normal domain with field of quotients F and residue field k. If A is R-central separable and $\mathrm{c\,soc}(A) = R$ then index $(A \otimes_R F) \geqslant \mathrm{index}(A \otimes_R k)$, where the index is the degree of the "division algebra part" of the algebra. Moreover, if index $(A \otimes_R F) = \mathrm{index}(A \otimes_R k)$, then $A \simeq B \otimes M_r(R)$ where $M_r(R)$ is the ring of $r \times r$ matrices over R and B is an R-central separable algebra with $B \otimes_R F$ a division algebra.

An interesting special case is where $A \otimes_R F \simeq M_n(F)$. The result (Theorem 3.1) is that $\mathrm{c\,soc}(A) = R$ if and only if $A \simeq M_n(R)$. One consequence of this (see the remarks after Corollary 3.2) is that if R is a normal domain (not necessarily local) and A is a central separable algebra with $A \otimes_R F \simeq M_n(F)$, then the set of points P in Spec(R) at which A is not split is a closed subset of the singular locus of R.

2. DEFINITION OF CLOSED SOCLE

In this section, in which the closed socle is defined, the conditions on R and A may be relaxed. Assume R is a normal domain, with field of quotients F, and A is a central separable R-algebra. Let $\Sigma = A \otimes_R F$. It is known that Σ is a central simple F-algebra. (For this and other basic properties of central separable algebras, see Auslander and Goldman [4], DeMeyer and Ingraham [7], or Orzech and Small [10].) Since A is R-central separable, A is in particular torsion free as an R-module. It follows that the canonical map $a \to a \otimes 1$ of A into Σ is an R-algebra monomorphism. We may and often will identify A with its image, an R-subalgebra of Σ. Under that identification, $AF = \Sigma$

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and since A is finitely generated as an R-module, A is an R-order in Σ, in the sense of Auslander and Goldman [3].

DEFINITION 2.1. A left ideal L of the central separable R-algebra A is *closed* if given $r \in R$, $r \neq 0$, and $a \in A$ such that $ra \in L$, it follows that $a \in L$. (In other words I is an R-pure submodule of A.)

Note that if L' is a left ideal of Σ, then $L = L' \cap A$ is a closed left ideal of A: L is clearly a left ideal of A and if $ra \in L$, $r \in R$, $r \neq 0$, and $a \in A$, then $ra \in L'$, so $a \in L'$. Hence $a \in L' \cap A = L$, as desired. In fact this is precisely the way closed left ideals come about, as the following lemma shows.

LEMMA 2.2. *There is a one to one, order preserving correspondence between the left ideals of Σ and the closed left ideals of A, given by $L \to L \otimes_R F$ $(=LF)$ for L a closed left ideal of A and $L' \to L' \cap A$ for L' a left ideal of Σ.*

Proof. Since LF is a left ideal of Σ and $L' \cap A$ was shown to be a closed left ideal of A, the maps are well defined. Each is clearly order preserving.

We show next that $L = LF \cap A$. Clearly $L \subset LF \cap A$. Let $x \in LF \cap A$. Then $x = \sum_i x_i f_i$ for some $x_i \in F$. Choose $r \in R$ such that $r \neq 0$ and $rf_i \in R$ for all i. Then $rx = \sum_i x_i(rf_i)$, so $rx \in L$ and $x \in A$. Thus $x \in L$ as desired. Hence $LF \cap A = L$.

Finally we need to show $(L' \cap A)F = L'$, for L' a left ideal of Σ. Clearly we have $(L' \cap A)F \subseteq L'$. Let $x \in L'$. Since A is an R-order in Σ, there exists $s \in R$ such that $sx \in A$. Let $a = sx$. Then $a \in L' \cap A$ and $x = s^{-1}a$, so $x \in (L' \cap A)F$. Hence $(L' \cap A)F \supseteq L'$ and we are done.

It follows from this lemma that A satisfies the descending chain condition on closed left ideals. In particular minimal closed (nonzero) left ideals exist in A.

PROPOSITION 2.3. *Let I be the sum of the minimal closed left ideals of A. Then I is a (two-sided) ideal of A.*

Proof. I is clearly a left ideal of A. To show I is a right ideal, let $a \in A$ and let L be a minimal closed left ideal of A. We need to show $La \subseteq I$. We may assume $La \neq 0$. Then $0 \neq (La)F = (LF)a$. By the lemma above LF is a minimal left ideal of Σ. It follows that $(LF)a$ is a minimal left ideal of Σ. Since $La \subseteq (LF)a \cap A$ and $(LF)a \cap A$ is a minimal closed left ideal of A, we conclude $La \subset I$, as desired.

Since A is a central separable R-algebra, it is known that there is a one-to-one correspondence between the ideals of A and the ideals of R, given by $B \to B \cap R$ for B an ideal of A and $T \to TA$ for T an ideal of R. In particular, with the notation of the previous proposition, we have $I = (I \cap R)A$.

DEFINITION 2.4. Let I be the sum of the minimal closed left ideals of A. The ideal $I \cap R$ of R is called the *closed socle* of A and is denoted c soc(A).

3. The Split Case

In this section an important special case of the main theorem will be proved. The result will be useful in proving the main theorem and has interesting consequences of its own.

Recall that a central separable R-algebra A is called *split* if $A \cong \mathrm{End}_R(P)$ for some finitely generated projective R-module P.

THEOREM 3.1. *Let R be a local normal domain with field of quotients F. Let A be a central separable R-algebra such that $A \otimes_R F \cong M_n(F)$, as F-algebras, for some n. Then $A \cong M_n(R)$ if and only if* $\mathrm{c\,soc}(A) = R$.

Before giving the proof of this result, some corollaries will be stated.

COROLLARY 3.2. *Let R be a normal domain, with field of quotients F. Suppose A is a central separable R-algebra, with $A \otimes_R F \sim 1$. Let P be a prime ideal of R. Then A_P $(=A \otimes_R R_P)$ is split if and only if $P \not\supseteq$* $\mathrm{c\,soc}(A)$.

Proof. The result follows easily from the following fact: $\mathrm{c\,soc}(A_P) = (\mathrm{c\,soc}(A))R_P$. To prove this equality, first note that if L is a closed left ideal of A, then LR_P is a closed left ideal of A_P. Also, if L' is a closed left ideal of A_P, then $L' = (L' \cap A)R_P$ and $L' \cap A$ is a closed left ideal of A. These facts follow from the proof of Lemma 2.2. It follows that

$$\Bigl(\sum_{L \text{ min closed in } A} L\Bigr) \cdot R_P = \Bigl(\sum_{L \text{ min closed in } A} LR_P\Bigr) = \sum_{L' \text{ min closed in } A_P} L'.$$

Intersecting with R_P, we obtain $(\mathrm{c\,soc}(A)) \cdot R_P = \mathrm{c\,soc}(A_P)$ as desired. Then by the theorem, $A_P \sim 1$ if and only if $\mathrm{c\,soc}(A_P) = R_P$. It follows that $A_P \sim 1$ if and only if $(\mathrm{c\,soc}(A))R_P = R_P$ which is equivalent to $P \not\supseteq \mathrm{c\,soc}(A)$.

It follows from this corollary that under its hypotheses the set T of points at which A is not split is a closed subset of $\mathrm{Spec}\, R$. If R is a regular local ring, A as above, then it is known that A is split (Auslander and Goldman [4]). It follows that in the general case (i.e., R a normal domain) T is a closed subset of $\mathrm{Spec}\, R$ sitting inside the singular locus of R.

The following is an immediate consequence of Corollary 3.2.

COROLLARY 3.3. *Let R and A be as in the previous corollary. Then A is split at every prime of R (i.e., $A_P \sim 1$ for all prime ideals of R) if and only if* $\mathrm{c\,soc}(A) = R$.

COROLLARY 3.4. *Suppose R and A are as in the previous corollary. Assume moreover that R is locally factorial, that is that R_P is a factorial ring for each prime ideal P of R. Then A is split if and only if* $\mathrm{c\,soc}(A) = R$.

Proof. It has been shown (Auslander [2]) that if R is locally factorial and A is locally split (i.e., $A_P \sim 1$ for all prime ideals P) then A is split. Hence the result follows from Corollary 3.3.

We now turn to the proof of Theorem 3.1. The following fact will be useful.

LEMMA 3.5. *Under the hypotheses of the theorem, let L be a minimal closed left ideal of A. If L is free as an R-module, then $A \simeq \mathrm{End}_R(L)$ as R-algebras, where the isomorphism $A \to \mathrm{End}_R(L)$ is left multiplication. In particular $A \simeq M_n(R)$.*

Proof. The map $A \to \mathrm{End}_R(L)$ is clearly an R-algebra homomorphism. It is a monomorphism, since the left annihilator of L in A is contained in the left annihilator of LF in Σ, and this latter is zero. Now $\mathrm{End}_R(L)$ is an order in $M_n(F)$. Since A is R-central separable, A is a maximal order in $M_n(F)$ (Auslander and Goldman [4]). Hence the map must be onto, so $A \simeq \mathrm{End}_R(L)$.

Proof of Theorem 3.1. We first show that if $A \simeq M_n(R)$, for some n, then $\mathrm{c\,soc}(A) = R$. If $A \simeq M_n(R)$, then A contains e_{ii}, where e_{ii} is the matrix with 1 in the (i, i) position and 0 in all other positions. But it is easy to see that Ae_{ii} is a minimal closed left ideal of A for each i, $i = 1,\ldots, n$. Hence the sum of the minimal closed left ideals of A contains $Ae_{11} + \cdots + Ae_{nn} = A$, so $\mathrm{c\,soc}(A) = R$.

Now assume $\mathrm{c\,soc}(A) = R$. Then the sum of the minimal closed left ideals of A must equal A. Let m be the maximal ideal of R. By the one-to-one correspondence between ideals of A and ideals of R, we have that mA is the radical of A and is the unique maximal ideal of A. Hence, by the assumption, there is a minimal closed left ideal L of A such that $L \not\subseteq mA$. We will eventually show that L is R-free.

Now consider $LF \subseteq M_n(F)$. By the minimality of LF, there is an F-algebra automorphism σ of $M_n(F)$ such that $\sigma(LF)$ is the first column of $M_n(F)$, that is $\sigma(LF) = M_n(F)e_{11}$. Now $\sigma(A)$ is a central separable R-algebra, isomorphic as an R-algebra to A. Also $\sigma(L) = \sigma(LF \cap A) = M_n(F)e_{11} \cap A$. It follows that by working with $\sigma(A)$ instead of A, we may assume $L = M_n(F)e_{11} \cap A$ and $L \not\subseteq mA$. L then consists of those elements of A with zero entries off the first column.

Since $L \not\subseteq mA$, and mA is the radical of A, it follows that there is an element $l \in L$ such that $1 - l$ is not a unit in A. (That is, L is not left quasi-regular.) Let

$$l = \begin{pmatrix} \alpha_1 & \\ \vdots & 0 \\ \alpha_n & \end{pmatrix} \qquad \text{where } \alpha_i \in F,\ i = 1,\ldots, n.$$

Since $l \in A$ and A is finitely generated as an R-module, l is integral over R. Writing down an integral equation for l shows that α_1 is then integral over R. Since $\alpha_1 \in F$ and R is normal, we conclude that $\alpha_1 \in R$.

We claim α_1 is a unit in R: If not, then $\alpha_1 \in m$, so $1 - \alpha_1$ is a unit in R. But it then follows that $1 - l$ is a unit in A: The characteristic polynomial of $1 - l$ is

$$\det(\lambda I - (1 - l)) = \det \begin{pmatrix} \lambda - (1 - \alpha_1) & 0 & & & 0 \\ \alpha_2 & \lambda - 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \alpha_n & 0 & & & \lambda - 1 \end{pmatrix}$$

$$= (\lambda - (1 - \alpha_1))(\lambda - 1)^{n-1}$$

which has coefficients in R and constant term $\pm(1 - \alpha_1)$. By the Cayley–Hamilton theorem, this gives an inverse for $1 - l$ in A. However, this is contrary to our assumption. We conclude α_1 is a unit in R.

Let $t = \alpha_1^{-1} l$. Then $t \in L$, and

$$t = \begin{pmatrix} 1 & \\ b_2 & \\ \vdots & 0 \\ b_n & \end{pmatrix},$$

some $b_i \in F$. It follows that for any $s \in L$, $st = s$. In particular $t^2 = t$, so t is an idempotent, and $Lt = L$. We conclude that $L = At$. Since $A = At \oplus A(1 - t)$ and A is R-free, it follows that L is R-projective, hence R-free. Using Lemma 3.5, the proof is complete.

4. Main Theorem

In this section the main result of the paper will be proved (Theorem 4.6). We will need several lemmas.

Lemma 4.1. *Let R be a normal domain. Let L be a minimal closed left ideal of $M_n(R)$ and let T be a left ideal of $M_n(R)$ such that $T \subseteq L$ and $TM_n(R) = M_n(R)$. Then $T = L$.*

Proof. It suffices to show $T_m = L_m$ for each maximal ideal m of R. Hence we may assume R is local, with maximal ideal m. The condition on T is then equivalent to $T \not\subseteq mM_n(R)$. It follows that $L \not\subseteq mM_n(R)$. Since L is minimal closed, we conclude by the proof of Theorem 3.1 that $L = M_n(R)e$ for some idempotent e.

Let "—" denote reduction mod m. Then $\bar{L} = M_n(\bar{R})\bar{e}$ is a minimal left ideal of $M_n(\bar{R})$. Since $\bar{T} \subseteq \bar{L}$ and $\bar{T} \neq 0$, we conclude that $\bar{T} = \bar{L}$. Hence $L \subseteq T + mM_n(R)$.

We *claim* that $L \subseteq T + mL$: If $l \in L$, then $l = t + y$, for some $t \in T$ and

$y \in mM_n(R)$. Now $L = M_n(R)e$ and $y = l - t \in L$, so $ye = y$. Hence $y \in mM_n(R)e = mL$. Hence $l \in T + mL$, as desired.

Since $T + mL \subseteq L$, we have $L = T + mL$. Hence by Nakayama's lemma, $L = T$ and we are done.

The idea for the proof of the next lemma was communicated to me by D. Zelinsky.

LEMMA 4.2. *If S is a semilocal domain and L is a left ideal of $M_n(S)$ such that $LM_n(S) = M_n(S)$, then there is a minimal idempotent e in $M_n(S)$ (i.e., e is minimal when considered in $M_n(F)$, where F is the field of quotients of S) such that $LeM_n(S) = M_n(S)$.*

Proof. We will in fact show there exists an invertible element a in $M_n(S)$ such that $Lae_{11}M_n(S) = M_n(S)$, where e_{11} is the usual matrix unit. The result follows with $e = ae_{11}a^{-1}$.

Let J be the Jacobson radical of S. Then $\bar{S} = S/J = F_1 \oplus \cdots \oplus F_r$, where each F_i is a field and $F_i = \bar{S}e_i$ for each i, where e_i is an idempotent. Then $\bar{e}_{11} = f_1 + \cdots + f_r$, where $f_i = e_i\bar{e}_{11}$ and no f_i is zero. Now $\bar{L}$ is a left ideal of $M_n(\bar{S})$ and $\bar{L}M_n(\bar{S}) = M_n(\bar{S})$. We have $\bar{L} = L_1 \oplus \cdots \oplus L_r$ where $L_i = e_iL$ is a left ideal in $M_n(\bar{S})e_i = M_n(F_i)$. Moreover, since $\bar{L}M_n(\bar{S}) = M_n(\bar{S})$, we have $L_iM_n(F_i) = M_n(F_i)$ for all i. Now the desired result is certainly true over fields, so there exists, for each i, an invertible element a_i in $M_n(F_i)$ such that $L_ia_if_iM_n(F_i) = M_n(F_i)$. Let $\bar{a} = \sum_{i=1}^r a_i$. Then $\bar{a}$ is invertible in $M_n(\bar{S})$ and $\bar{L}\bar{a}\bar{e}_{11}M_n(\bar{S}) = \sum L_ia_if_iM_n(F_i) = \sum M_n(F_i) = M_n(\bar{S})$. It follows that a is invertible in $M_n(S)$ and $\overline{Lae_{11}M_n(S)} = \bar{L}\bar{a}\bar{e}_{11}M_n(\bar{S}) = M_n(\bar{S})$. By Nakayama's lemma, $Lae_{11}M_n(S) = M_n(S)$ and we are done.

COROLLARY 4.3. *Under the conditions of the lemma, L contains a minimal idempotent.*

Proof. By the lemma, there is a minimal idempotent e in $M_n(S)$ such that $LeM_n(S) = M_n(S)$. But $Le \subseteq M_n(S)e$, so it follows by Lemma 4.1 that $Le = M_n(S)e$. Since $M_n(S)e$ is an $M_n(S)$-direct summand of $M_n(S)$, we have that $M_n(S)e = Le$ is projective as a left $M_n(S)$-module. Hence the following exact sequence splits:

$$0 \to \ker \to L \to Le \to 0.$$

Now $e \in Le$ and if $\varphi: Le \to L$ denotes a splitting map for this sequence then an easy computation shows that $\varphi(e)$ is an idempotent. This proves the lemma.

Recall that if F is a field and A is a central simple F-algebra, then by Wedderburn theory $A \cong D \otimes M_r(F)$ for a uniquely determined F-central division algebra D and integer r. If $[D : F] = s^2$, then sr is the *degree* of A and s is the *index* of A. If L is a left ideal of A, then $[L : F]/s^2r$ is called the *rank* of L.

DEFINITION 4.4. If L is a left ideal in the central separable R-algebra A, R an integral domain, then the *rank* of L is the rank of the left ideal $L \otimes_R F$ in $A \otimes_R F$, where F is the field of quotients of R.

LEMMA 4.5. *Let S be a semilocal domain and L a left ideal in $M_n(S)$. For each maximal ideal N of S, we have rank$(L) \geqslant$ rank$(\bar{L})$, where "$-$" denotes reduction* mod N. *Moreover, if for each maximal ideal N of S we have rank$(L) =$ rank$(\bar{L})$, then $L = M_n(S)e$, for some idempotent e.*

Proof. For the first statement, we may assume S is local with maximal ideal N. We proceed by induction on the rank of L:

If $L \subseteq NM_n(S)$, then $\bar{L} = 0$ and we are done.

Hence we may assume $L \not\subseteq NM_n(S)$, so that $LM_n(S) = M_n(S)$. By Corollary 4.3, L contains a minimal idempotent e, so $L = L' \oplus M_n(S)e$, where $L' = \{x \in L \mid xe = 0\}$. Now rank$(M_n(S)e) =$ rank$(M_n(\bar{S})\bar{e}) = 1$. Also rank$(L) =$ rank$(L') +$ rank$(M_n(S)e)$. By induction rank$(L') \geqslant$ rank$(\bar{L}')$. Since $\bar{L} = \bar{L}' + M_n(\bar{S})\bar{e}$, we conclude rank$(L) =$ rank$(L') + 1 \geqslant$ rank$(\bar{L}') + 1 \geqslant$ rank$(\bar{L})$, and we are done.

To prove the second statement, we again proceed by induction on the rank of L. We may assume $L \neq 0$. It follows that for each maximal ideal N of S, we have $L \not\subseteq NM_n(S)$. Hence $LM_n(S) = M_n(S)$. By Corollary 4.3 we have $L = L' \oplus M_n(S)e$ for some minimal idempotent e, where $L' = \{x \in L \mid xe = 0\}$. For each maximal ideal N of S we have $\bar{L} = \bar{L}' + M_n(\bar{S})\bar{e}$, where "$-$" denotes reduction mod N. It follows that $1 +$ rank$(L') =$ rank$(L) =$ rank$(\bar{L}) \leqslant$ rank$(\bar{L}') + 1$. Since by the first half of the theorem rank$(L') \geqslant$ rank$(\bar{L}')$, we conclude rank$(L') =$ rank$(\bar{L}')$. Applying induction, we have $L' = M_n(S)f$, for some idempotent f. But then $fe = 0$, so e and $f - ef$ are orthogonal idempotents. Since $M_n(S)f = M_n(S)(f - ef)$, we have $L = M_n(S)(e + (f - ef))$ and we are done.

We can now proceed to the main theorem.

THEOREM 4.6. *Let R be a local normal domain, with field of quotients F and maximal ideal m. Let A be a central separable R-algebra with* c soc$(A) = R$. *Then* index$(A \otimes_R F) \geqslant$ index(A/mA). *Moreover, if* index$(A \otimes_R F) =$ index(A/mA), *then $A \simeq B \otimes M_r(R)$, for some integer r, where B is an R-central separable algebra such that $B \otimes_R F$ is a division algebra.*

Proof. Let $[A : R] = n^2$. Let $s =$ index$(A \otimes_R F)$ and $k =$ index(A/mA). Since c soc$(A) = R$, there is a minimal closed left ideal L of A such that $L \not\subseteq mA$.

There is a splitting ring S of A (i.e., a ring S containing R such that $A \otimes_R S \simeq M_n(S)$) such that S is a semilocal normal domain and such that S is free and finitely generated as an R-module (see Orzech and Small [10, p. 124 (11.3)] and Janusz [9, p. 473 (4.3)]). Since L has rank one and index$(A \otimes_R F) = s$, it follows that rank$(L \otimes_R S) = s$.

Let N be a maximal ideal of S and let "′" denote reduction mod N. We have the following commutative diagram, where "—" denotes reduction mod m:

$$\begin{tikzcd} A \otimes_R S = M_n(S) \arrow[dr, "\prime"] & \\ A \arrow[u] \arrow[dr, "-"'] & M_n(S') = A \otimes_R S' = \bar{A} \otimes_R S'. \\ & \bar{A} = A/mA \arrow[u] \end{tikzcd}$$

In particular S' splits $\bar{A}$. Since $\operatorname{rank}(\bar{L}) \geqslant 1$, we conclude $\operatorname{rank}(\bar{L} \otimes_{\bar{R}} S') \geqslant k$. Also it is clear that $(L \otimes_R S)' = L \otimes_R S' = \bar{L} \otimes_{\bar{R}} S'$, so $\operatorname{rank}((L \otimes_R S)') \geqslant k$. By Lemma 4.5, we know $\operatorname{rank}(L \otimes S) \geqslant \operatorname{rank}((L \otimes S)')$, so $s \geqslant k$ as desired.

Moreover, if $s = k$ then we have $s = \operatorname{rank}(L \otimes S) \geqslant \operatorname{rank}((L \otimes S)') \geqslant k = s$, so $\operatorname{rank}(L \otimes S) = \operatorname{rank}(L \otimes S)')$. Since the maximal ideal N was chosen arbitrarily we conclude that for each maximal ideal N of S, $\operatorname{rank}(L \otimes S) = \operatorname{rank}((L \otimes S)')$. Again by Lemma 4.5, we conclude $L \otimes_R S = M_n(S)e$, for some idempotent e. In particular $L \otimes_R S$ is S-free. Since S is R-free, we have $L \otimes_R S \cong L \oplus \cdots \oplus L$ ($[S : R]$ copies) as R-modules. It follows that L is R-free.

We have the canonical map $A \to \operatorname{End}_R(L)$ given by left multiplication and this is an R-algebra monomorphism. We will identify A with its image in $\operatorname{End}_R(L)$. Now $\operatorname{End}_R(L)$ is R-central separable, so $\operatorname{End}_R(L) = A \otimes_R C(A)$, where $C(A)$ is the centralizer of A in $\operatorname{End}_R(L)$. Hence $[A] = [C(A)^0]$ in the Brauer group of R.

Now let $A \otimes_R F = D \otimes M_r(F)$ where D is an F-central division algebra and $r = n/s$. Then $\operatorname{End}_R(L) \otimes F = \operatorname{End}_F(LF) = (D \otimes M_r(F)) \otimes D^0$. Hence we have $A \otimes_R C(A) \otimes_R F = (D \otimes M_r(F)) \otimes D^0$, $A \otimes F = D \otimes M_r(F)$, and $C(A)$ centralizes A. It follows that $C(A) \otimes_R F = D^0$. Letting $B = C(A)$, we have $[A] = [B^0]$ in the Brauer group of R and $B \otimes_R F$ is a division algebra. Since B contains no proper idempotents and R is local, it follows from a theorem of DeMeyer [6, p. 40] that $A \cong B^0 \otimes M_r(R)$, so we are done.

Note that Theorem 3.1 is a special case of Theorem 4.1: If $\operatorname{index}(A \otimes_R F) = 1$, then by the first half of the theorem we have $\operatorname{index}(A/mA) \leqslant 1$. It follows that $\operatorname{index}(A/mA) = 1$ and so by the second half of the theorem $A \cong M_n(R)$. We also have the following result:

Corollary 4.7. *Suppose R and m are as above and A is an R-central separable algebra with A/mA a division algebra. Then $A \otimes_R F$ is a division algebra if and only if* c $\operatorname{soc}(A) = R$.

Proof. If $A \otimes_R F$ is a division algebra, then it is clear that c soc$(A) = R$. Conversely, if c soc$(A) = R$, then by the theorem index$(A \otimes_R F) \geqslant$ index(A/mA). Now as an R-module A is free of rank n^2, for some n. Then $[A/mA : R/m] = n^2$, so index$(A/mA) = n$. Hence index$(A \otimes_R F) \geqslant n$. But $[A \otimes_R F : F] = [A : R] = n^2$, so index$(A \otimes_R F) = n$ and $A \otimes_R F$ is a division algebra.

5. Example

We want to give an example where the closed socle can be computed. Let k be a field of characteristic $\neq 2$. Let

$$R = \frac{k[x, y, z]_{(x,y,z)}}{(z^2 - \alpha x^2 - \beta y^2)}$$

where α and β are units in $k[x, y, z]_{(x,y,z)}$. It is shown in Fossum [8, p. 50] that R is a local normal domain. Over R we can form a quaternion algebra (α, β), that is the four-dimensional R-algebra with basis 1, u, v, uv and defining relations $u^2 = \alpha$, $v^2 = \beta$, $uv = -vu$. It is known that (α, β) is a central separable R-algebra (Orzech and Small [10, p. 21]). If F is the field of quotients of R, then in F we have $1 = \alpha(x/z)^2 + \beta(y/z)^2$. It follows that $(\alpha, \beta) \otimes_R F \cong M_2(F)$. Hence (α, β) is the type of algebra we discussed in Section 3. These algebras are examined at length in Childs *et al.* [5]. We want to determine c soc$((\alpha, \beta))$. Let $\bar{x}, \bar{y}, \bar{z}$ be the images of x, y, z in R.

THEOREM 5.1. c soc$((\alpha, \beta)) \supseteq (\bar{x}, \bar{y}, \bar{z})$, *the maximal ideal of R. In particular, if (α, β) is not split, then* c soc$((\alpha, \beta)) = (\bar{x}, \bar{y}, \bar{z})$.

Proof. We may identify (α, β) with an R-subalgebra of $M_2(F)$. Since all proper left ideals of $M_2(F)$ are minimal, it follows that all proper closed left ideals of (α, β) are minimal. In particular, if $a \in (\alpha, \beta)$ is a zero divisor then a is contained in a minimal left ideal of $M_2(F)$, hence in a minimal closed left ideal of A. Thus any element a in (α, β) with $\det(a) = 0$ is in the sum of the minimal closed left ideals of (α, β).

Let $a \in (\alpha, \beta)$, $a = a_1 + a_2u + a_3v + a_4uv$, $a_i \in R$. Then $\det(a) = a_1^2 - a_2^2\alpha - a_3^2\beta + a_4^2\alpha\beta$. Consider the choice $a_1 = \bar{z}$, $a_2 = \bar{x}$, $a_3 = \bar{y}$, $a_4 = 0$ and $a_1 = \bar{z}$, $a_2 = -\bar{x}$, $a_3 = -\bar{y}$, $a_4 = 0$. Both of these give zero divisors in (α, β). It follows that $\bar{z} + \bar{x}u + \bar{y}v$ and $\bar{z} - \bar{x}u - \bar{y}v$ are in closed left ideals of (α, β), so $2\bar{z}$ is in c soc$((\alpha, \beta))$. Hence $\bar{z} \in$ c soc$((\alpha, \beta))$. Similar arguments show that $\bar{x}, \bar{y}$ are in c soc$((\alpha, \beta))$, so we are done.

It is known that if $k = R$, $\alpha = -1$, $\beta = -1$, then the algebra $(-1, -1)$ is not split. Hence the theorem implies that c soc$((-1, -1)) = (\bar{x}, \bar{y}, \bar{z})$ in this case.

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